MATH Math 162A Review: Abstract Vector Space

1. We define

$$V = \left\{ \sum_{k=0}^{\infty} a_k x^k \mid a_k \in \mathbb{R}, \forall k \in \mathbb{Z} \right\}$$

be the set of all *formal* power series (in the sense that we don't consider its convergence). Then V is a vector space.

Let

$$V_1 = \{ f \in V \mid \text{the series is absolutely convergent} \}$$

Then V_1 is a subspace of V.

Solution: We first define the binary operations $(+, \cdot)$

Define: ① $\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

② $r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (r a_k) x^k$

The next step would be tedious: we need to check that all the eight identities are correct. For example, we need to prove

$$\left(\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k \right) + \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} a_k x^k + \left(\sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} c_k x^k \right)$$

but both sides are equal to

$\sum_{k=0}^{\infty} (a_k + b_k + c_k) x^k$ so they are equal

I will omit the rest of proofs.

Now we prove that V_1 is a subspace of V, hence V_1 is a vector space itself. Note that this time we <u>do not</u> need to check the eight identities. We just need to prove that the binary operations are <u>closed</u> in V_1. That is, the addition of two absolutely convergent series is still absolutely convergent, and the scalar multiplication of absolutely convergent series is still absolutely convergent —— both of them are true by Calculus.

2. Prove that the functions $\{e^x, \sin x, \cos x\}$ are linearly independent.

Solution: We solve the equation

$$c_1 e^x + c_2 \sin x + c_3 \cos x = 0$$

Since the above equation is true for $\forall x$, we take $x=0$ to get $c_1 + c_3 = 0$. Now we let $x = \pi$. Then we get

$$c_1 e^{\pi} - c_3 = 0 \Rightarrow c_1(e^{\pi}+1) = 0 \Rightarrow c_1 = 0$$

$$\Rightarrow c_3 = 0 \Rightarrow c_2 = 0.$$

Thus the three functions are L.I.